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Classifying the general family of 1D point interactions: a scattering approach

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Abstract

We propose a classification scheme for the complete family of 1D point interactions. To do so, we first review the solutions of the wavefunctions and Green functions of the problem. Second, we derive the exact time-dependent propagators in such a way that we can write the expressions for the K's in a very compact form. As they should, such expressions do reduce to the known results in the literature according to the potential parameter values. Then, we analyse in general terms how the different point interactions scatter off arbitrary initially localized wave packets. Finally, we show that the physical features associated with the scattering process can be used to establish a classification procedure. Moreover, these physical characteristics are directly related to the potential parameters leading to the many formulae for the K's. As an application, we present numerical calculations for Gaussian wave packets.

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1. Introduction

Few classes of potentials in quantum mechanics present, at the same time, the very welcome features of being useful to study different phenomena, and allowing exact closed analytical solutions in a large number of situations. The general family of one-dimensional (1D) point interactions is a good example of that. Physically, they can be faced as a singular one-point potential of support, say, at $x = 0$, thus otherwise identically null along the line [1]. Mathematically, point interactions can be treated through a self-adjoint extension of the 1D kinetic energy Hamiltonian operator [2].

From a practical point of view, these short-range potentials are used to model many interesting problems [3], for instance, duality between bosons and fermions [4], quasi-one-dimensional bose gases [5], different aspects in many-body interactions [6], Josephson junctions [7], wave packet revival [8], quantum graphs [9], the implementation of quantum

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abacus [10], etc. Furthermore, since point interactions can be approximated by well-behaved smooth potentials [11] (even in the relativistic case [12]), in principle they are realizable under actual laboratory conditions, open a great avenue for applications.

Analysis of point interactions in a more formal sense, i.e., broadly viewed as a family of regularized potentials characterized by a set of parameters, on the other hand are mainly focused on their construction and generalization. For such, besides self-adjoint extensions other methods are also employed, like functional analysis and path integrals formalism [13, 14], global $U(2)$ group description [4] or higher order derivatives as boundary conditions [15].

However, as far as we know there are no works dealing with a possible classification for these potentials, from which, for example, one could distinguish different physical properties from different point interactions. It is true that a kind of mathematical classification emerges from a very nice work [16] (see also [17]), which addresses how the analytical expressions for the propagators K depend on the potential parameter values. But no further interpretation is carried out. Our aim in this contribution is to propose a general classification scheme for non-relativistic arbitrary point interaction potentials. To do so, we first review the problem's complete solution for the wavefunctions and the energy Green functions. Then, we derive their exact time-dependent propagators by considering an approach which does lead to very compact formulae for the K's of all possible cases, hence summarizing the previous calculations in the literature. Finally, we show how to identify different features for the potentials in terms of the way they scatter off wave packets, also making a connection with the mathematical results in the first part of the paper. We should mention that the very intriguing aspects of transmission through and reflection from point interactions have been discussed in some particular contexts, as, time-dependent potentials [18], nonlinear Schrödinger equation [19] and shredding by sparse barriers [20]. Nevertheless, here a general systematic analysis of scattering by point interactions is discussed for the first time.

2. General point interactions

2.1. The problem formulation and the wavefunction solutions

In a simple heuristic picture, an arbitrary point interaction can be regarded as a zero-range potential, thus having a single-point support, which here we assume to be the origin $\{0\}$. Therefore, the problem can be formulated as the free Schrödinger operator $H_0(x) = -\frac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2$ (defined on the line, i.e., for $-\infty < x < +\infty$) 'perturbed' by such potential.

In a more rigorous construction, the full Hamiltonian H of the system can be viewed as the self-adjoint extension of the operator $H_0|_{C_0^\infty(\mathbb{R}\setminus\{0\})}$. As discussed in the introduction, there are different mathematical approaches to treat the problem. In particular, a complete and deep analysis is presented in [2]. From the standard theory of linear operators in Hilbert space (see, for instance, the textbook [21] or the didactic review in [22]) we find that $H_0|_{C_0^\infty(\mathbb{R}\setminus\{0\})}$ has deficiency indices $(2, 2)$, thus leading to a family of four real parameters of possible extensions H.

It is important to observe that when we apply self-adjoint extension techniques to a Hamiltonian operator, we are in fact ensuring current density conservation [23]. Hence, the above-mentioned four parameters, coming from the self-adjoint extensions, are directly related to the boundary conditions that the wavefunction φ must satisfy at the potential location $x = 0$. Then, the original problem is mathematically equivalent to the following [2]: to solve $H_0(x)\varphi(x) = \frac{1}{2}k^2\varphi(x)$ on the line, for φ subjected to

$$\begin{pmatrix} \varphi(0^+) \\ \varphi'(0^+) \end{pmatrix} = \omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi(0^-) \\ \varphi'(0^-) \end{pmatrix}. \tag{1}$$

Here, a, b, c and d are real with $ad - bc = 1$ and $|\omega| = 1$. So, we have specific point interactions by choosing specific values for the parameters. For instance, if $a = d = \omega = 1$ we get from $b = 0, c = 2\lambda$ and from $c = 0, b = 2\lambda$, respectively, the δ and δ' potentials of strength λ.

We should observe that it has been proven for point interactions [24, 25] that generally the on-shell S-matrix at a single energy determines the boundary conditions. It allows one to characterize such potentials in terms of their scattering properties, what is totally equivalent to imposing (1) (see, e.g., [26]). This feature is one of our guidelines throughout the present work.

The scattering solution for the general condition (1), represented by a plane wave of wave number k and incident from either the left (+) or right (−), is written as [26]

$$\varphi_k^{(\pm)}(x) = \frac{1}{\sqrt{2\pi}} \times \begin{cases} \exp[\pm \mathrm{i}kx] + R^{(\pm)} \exp[\mp \mathrm{i}kx], & x \leqslant 0 \\ T^{(\pm)} \exp[\pm \mathrm{i}kx], & x \geqslant 0 \end{cases}, \tag{2}$$

where the scattering amplitudes are given by [26, 27]

$$R^{(\pm)} = \frac{c \pm \mathrm{i}k(d-a) + bk^2}{-c + \mathrm{i}k(d+a) + bk^2}, \qquad T^{(\pm)} = \frac{2\,\mathrm{i}k\omega^{\pm 1}}{-c + \mathrm{i}k(d+a) + bk^2}. \tag{3}$$

For a latter purpose it is quite useful to decompose the coefficients (3) into partial fractions. Then, for $b = 0$ we have ($\kappa_0 = -cd/(d^2+1)$)

$$R^{(\pm)} = \pm\left(\frac{d^2-1}{d^2+1}\right) + \left(\frac{2d^{(1\pm 1)}}{d^2+1}\right)\frac{\mathrm{i}\kappa_0}{k - \mathrm{i}\kappa_0}, \qquad T^{(\pm)} = \left(\frac{2d\omega^{\pm 1}}{d^2+1}\right)\left(1 + \frac{\mathrm{i}\kappa_0}{k - \mathrm{i}\kappa_0}\right), \tag{4}$$

whereas for $b \neq 0$ we find ($\kappa_\pm = -[(d+a) \pm \sqrt{(d-a)^2+4}]/(2b)$ and $\sigma(\kappa_\pm) = \mp 1$)

$$\begin{aligned} R^{(\pm)} &= 1 + \frac{2\,\mathrm{i}}{\sqrt{(d-a)^2+4}} \sum_{\kappa=\kappa_+,\kappa_-} \sigma(\kappa) \frac{(a+b\kappa)^{\mp 1}\kappa}{k - \mathrm{i}\kappa}, \\ T^{(\pm)} &= \frac{2\,\mathrm{i}\omega^{\pm 1}}{\sqrt{(d-a)^2+4}} \sum_{\kappa=\kappa_+,\kappa_-} \sigma(\kappa) \frac{\kappa}{k - \mathrm{i}\kappa}. \end{aligned} \tag{5}$$

Depending on the parameter values, the problem also admits bounded solutions. It is a well-known fact that any pole for the scattering matrix in the upper-half of the complex k plane (along the imaginary axis) represents an eigenvalue [28]. In our problem, such condition corresponds to the existence of positive roots for $b\kappa^2 + \kappa(d+a) + c = 0$, where we identify $\kappa = -\mathrm{i}k$. Thus, the system has: (i) one bound state when $b = 0$ and $\kappa_0 > 0$; (ii) two (one) bound states when $b \neq 0$ and both (just one) $\kappa_\pm$ are greater than 0. Furthermore, regarding (ii), if $ad = 1$ (i.e., $c = 0$) either κ_+ or κ_- is necessarily null. So, there is at most one bound state if $b \neq 0$ and $ad = 1$.

For $\kappa > 0$ representing one of the above-discussed cases, the corresponding bound state solution is ($\sigma(\kappa_0) = \mathrm{sign}(d)$, with sign(·) the signal function)

$$\varphi_\kappa(x) = \sqrt{\frac{2\kappa(a+b\kappa)\sigma(\kappa)}{\sqrt{(d-a)^2+4}}} \times \begin{cases} \exp[-\kappa x], & x > 0 \\ (\omega(a+b\kappa))^{-1} \exp[\kappa x], & x < 0. \end{cases} \tag{6}$$

Finally, we observe that the exact Green functions for general point interactions have been calculated in [26, 29]. By defining G_{+-} for $x > 0 > x_0$, G_{-+} for $x_0 > 0 > x$, G_{++} for $x, x_0 > 0$ and G_{--} for $x, x_0 < 0$, we have (with R and T given by (3))

$$\begin{aligned} G_{\pm\mp}(x, x_0; k) &= \frac{1}{\mathrm{i}k} T^{(\pm)} \exp[\mathrm{i}k|x - x_0|], \\ G_{\mp\mp}(x, x_0; k) &= \frac{1}{\mathrm{i}k} [\exp[\mathrm{i}k|x - x_0|] + R^{(\pm)} \exp[\mathrm{i}k(|x| + |x_0|)]]. \end{aligned} \tag{7}$$

The above formula is correct even if the system has bound states [26]. If we define $s = -\mathrm{sign}(x)$ and $s_0 = -\mathrm{sign}(x_0)$, we can regroup (7) into the single expression

$$G(x, x_0; k) = \frac{1}{\mathrm{i}k}\Bigg[\exp[\mathrm{i}k|x - x_0|] + \left(R^{(s_0)} + \frac{(1 - ss_0)}{2}(T^{(s_0)} - R^{(s_0)} - 1)\right)\exp[-\mathrm{i}k(sx + s_0x_0)]\Bigg]. \tag{8}$$

2.2. *The exact propagators*

The propagators for general point interactions have been discussed in [16, 17]. Here, however, we re-derive them in a different fashion. Our aim at doing so is twofold: first, to obtain more concise expressions for the K's, thus summarizing all the previous results in the literature; second, to allow a straightforward way to link the different mathematical solutions for such a class of potentials to the physical classification scheme we develop in section 3.

As is well known, the propagator is given in terms of the bound (bs) and scattering (ss) states by

$$K(x, t; x_0, 0) = \sum_{\kappa} \varphi_{\kappa}^{(\mathrm{bs})}(x)\varphi_{\kappa}^{(\mathrm{bs})^*}(x_0)\exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right] + \int \mathrm{d}k\, k\varphi_k^{(\mathrm{ss})}(x)\varphi_k^{(\mathrm{ss})^*}(x_0)\exp\left[-\mathrm{i}\frac{k^2}{2}t\right]. \tag{9}$$

Due to the particular form of (2) and (8) (see a discussion in [26]), and from the bound state solutions (6), existing only when $\kappa > 0$, we can write

$$K(x, t; x_0, 0) = \frac{2\omega^{(s_0-s)/2}}{\sqrt{(d-a)^2+4}}\sum_{\kappa}\frac{\kappa\sigma(\kappa)}{(a+b\kappa)^{(s+s_0)/2}}\theta(\kappa)\exp[\kappa(sx + s_0x_0)]\exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right] + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\, kG(x, x_0; k)\exp\left[-\mathrm{i}\frac{t}{2}k^2\right], \tag{10}$$

with $\theta(.)$ the Heaviside function. Obviously, $\sum_{\kappa}$ runs over $\kappa_{\pm}$ (κ_0) for $b \neq 0$ ($b = 0$).

Now, from (4)–(5) we see that the R and T dependences on k are given only by terms of the form $1/(k + \mathrm{i}\nu)$. Hence, we obtain the propagator (10) from integrals which are either Gaussians-like or of the type (ν and v reals)

$$I(\nu, v) = \int_{-\infty}^{+\infty}\mathrm{d}k\frac{1}{k + \mathrm{i}\nu}\exp[-\mathrm{i}(k - v)^2]. \tag{11}$$

For $\nu > 0$, we have [30]

$$I(\nu, v) = -\mathrm{i}\pi\exp[-\mathrm{i}(v + \mathrm{i}\nu)^2]\mathrm{erfc}\left[\frac{v + \mathrm{i}\nu}{\sqrt{\mathrm{i}}}\right]. \tag{12}$$

The case $\nu < 0$ is easily derived from the relation $I(-\nu, v) = -I(\nu, -v)$. So, with the help of $\mathrm{erfc}[-z] = 2 - \mathrm{erfc}[z]$ [30], we find for an arbitrary ν that

$$I(\nu, v) = 2\pi\,\mathrm{i}\exp[-\mathrm{i}(v + \mathrm{i}\nu)^2]\theta(-\nu) - \mathrm{i}\pi\exp[-\mathrm{i}(v + \mathrm{i}\nu)^2]\mathrm{erfc}\left[\frac{v + \mathrm{i}\nu}{\sqrt{\mathrm{i}}}\right]. \tag{13}$$

The above formula for $I(\nu, v)$ has a remarkable consequence. Because of the particular form of (10), it may seem that the analytical expression for the propagator is different if the point interaction does or does not possess bound states. However, similarly to what happens in the calculations for the Green functions [26, 31], the first term on the rhs of (13), present only when there are bound states, exactly cancels out the first term on the rhs of (10) (see the appendix). Therefore, the functional analytical expression for the propagator of a given point

interaction is the same regardless if the potential admits or not bound state solutions. Such a result has already been proved for the usual δ function [32], and in fact is valid in general. As a simple example, consider the δ' potential, which as previously mentioned, is characterized by the free parameter $b = 2\lambda$, so that $\kappa_+ = -1/\lambda, \kappa_- = 0$. Thus, if $\lambda < 0$, the system has one bound state. The analytical formula for K, nevertheless, is exactly the same for any value of λ (see section 2.3). We should mention that this feature for K is true for a large number of different potentials, not only for point interactions. The reasons for this are discussed, for instance, in [31].

Regrouping the results in the appendix, we have the final exact propagators

$$K(x,t;x_0,0) = \omega^{(s_0-s)/2} d^{(1-ss_0)/2} \left[K_0(x,t;x_0,0) + \left(\frac{d^2-1}{d^2+1}\right) \frac{(s_0 s + s_0 + s - 1)}{2} \times K_0(x,t;-ss_0x_0,0) \right] + \omega^{(s_0-s)/2} d^{(s_0+s)/2} \left(\frac{d\kappa_0}{d^2+1}\right) \times \exp[\kappa_0(sx+s_0x_0)] \exp\left[\mathrm{i}\frac{\kappa_0^2}{2}t\right] \operatorname{erfc}\left[-\sqrt{\frac{\mathrm{i}t}{2}}\kappa_0 - \frac{(sx+s_0x_0)}{\sqrt{2\,\mathrm{i}t}}\right], \tag{14}$$

for $b = 0$, and

$$K(x,t;x_0,0) = K_0(x,t;x_0,0) + ss_0 K_0(x,t;-ss_0x_0,0) + \frac{\omega^{(s_0-s)/2}}{\sqrt{(d-a)^2+4}} \sum_{\kappa=\kappa_+,\kappa_-} \frac{\kappa\sigma(\kappa)}{(a+b\kappa)^{(s_0+s)/2}} \times \exp[\kappa(sx+s_0x_0)] \exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right] \operatorname{erfc}\left[-\sqrt{\frac{\mathrm{i}t}{2}}\kappa - \frac{(sx+s_0x_0)}{\sqrt{2\,\mathrm{i}t}}\right], \tag{15}$$

for $b \neq 0$.

In a *tour de force* [16], different formulae for the propagator of a general point interaction are derived according to the values of the parameters a, b, c and d. In our framework, such many expressions can be viewed as a consequence of the different ways the quantum amplitudes R and T are decomposed into partial fractions. However, here we are able to write down single expressions for the K's by using the artefact of summing over the κ's. Also, our procedure makes clear why the functional form of the propagators does not change if the potentials allow bound states. Furthermore, as it should be, our results are completely analogous to those in [16]. Indeed, in [16] the final expressions for the propagators are obtained up to the calculation of some integrals, which once performed, lead to terms of the form exp[.] × erfc[.]. So, it is very lengthy but straightforward to show that our general equations (14) and (15) reproduce all the cases in [16].

2.3. *Examples*

To illustrate the above results, we calculate four particular cases: (i) $a = d = \omega = 1, b = 0$ and $c = 2\lambda$; (ii) $a = d = \omega = 1, b = 2\lambda$ and $c = 0$; (iii) $a = d = 0, \omega = 1, b = \lambda$ and $c = -\lambda^{-1}$ and (iv) $a^{-1} = d = \lambda, \omega = -\mathrm{i}, b = 0$ and $c = \lambda^{-1}$. As already stated, the first two are the usual δ and δ' potentials, respectively. Case (iii) corresponds to the boundary conditions $\varphi(x = 0^+) = \lambda\varphi'(x = 0^-)$ and $\varphi'(x = 0^+) = -\lambda^{-1}\varphi(x = 0^-)$, which we call the *crossed* case, since the value of φ in one side depends solely on the value of its derivative while in the other side of the point interaction. Finally, we call (iv) the *asymmetric* case [8] because by inserting its parameter values into (3), one finds different quantum amplitudes from the left

and the right. So, after a little algebra we have from (8) and (4)–(5) that

$$G^{\delta}(x,x_0;k)=\frac{1}{\mathrm{i}k}\left\{\exp[\mathrm{i}k|x-x_0|]+\left(\frac{-\mathrm{i}\lambda}{k+\mathrm{i}\lambda}\right)\exp[\mathrm{i}k(|x|+|x_0|)]\right\}$$

$$G^{\delta'}(x,x_0;k)=\frac{1}{\mathrm{i}k}\left\{\exp[\mathrm{i}k|x-x_0|]+\left(1+\frac{-\mathrm{i}\lambda^{-1}}{k+\mathrm{i}\lambda^{-1}}\right)ss_0\exp[\mathrm{i}k(|x|+|x_0|)]\right\}$$

$$G^{\mathrm{cros}}(x,x_0;k)=\frac{1}{\mathrm{i}k}\left\{\exp[\mathrm{i}k|x-x_0|]+\left[\frac{\mathrm{i}\lambda^{-1}}{k-\mathrm{i}\lambda^{-1}}+\left(1+\frac{-\mathrm{i}\lambda^{-1}}{k+\mathrm{i}\lambda^{-1}}\right)ss_0\right]\times\exp[\mathrm{i}k(|x|+|x_0|)]\right\}$$

$$G^{\mathrm{asym}}(x,x_0;k)=\frac{1}{\mathrm{i}k}\left\{\exp[\mathrm{i}k|x-x_0|]+\left[\frac{(-\mathrm{i}\lambda)^{(1+s_0)/2}}{1-\mathrm{i}\lambda}\frac{-\mathrm{i}(\lambda^2+1)^{-1}}{k+\mathrm{i}(\lambda^2+1)^{-1}}+\frac{(-\mathrm{i}\lambda)^{(1-s_0)/2}}{\mathrm{i}\lambda-1}+\frac{(\mathrm{i}\lambda)^{(1+s_0)/2}}{\mathrm{i}\lambda+1}\left(1+\frac{-\mathrm{i}(\lambda^2+1)^{-1}}{k+\mathrm{i}(\lambda^2+1)^{-1}}\right)ss_0\right]\times\exp[\mathrm{i}k(|x|+|x_0|)]\right\}. \tag{16}$$

Then, from equations (14) and (15), we find $[K_0(x,t;x_0,0)=K_0(x-x_0)]$

$$K^{\delta}(x,t;x_0,0)=K_0(x-x_0)+K_{\mathrm{int}}(|x|+|x_0|,\lambda)$$

$$K^{\delta'}(x,t;x_0,0)=K_0(x-x_0)+ss_0K_0(|x|+|x_0|)+ss_0K_{\mathrm{int}}(|x|+|x_0|,1/\lambda)$$

$$K^{\mathrm{cros}}(x,t;x_0,0)=K_0(x-x_0)+ss_0K_0(|x|+|x_0|)+K_{\mathrm{int}}(-(|x|+|x_0|),1/\lambda)+ss_0K_{\mathrm{int}}(|x|+|x_0|,1/\lambda)$$

$$K^{\mathrm{asym}}(x,t;x_0,0)=K_0(x-x_0)+\left(\frac{(-\mathrm{i}\lambda)^{(1-s_0)/2}}{\mathrm{i}\lambda-1}+\frac{(\mathrm{i}\lambda)^{(1+s_0)/2}}{\mathrm{i}\lambda+1}ss_0\right)K_0(|x|+|x_0|)+\left(\frac{(-\mathrm{i}\lambda)^{(1+s_0)/2}}{1-\mathrm{i}\lambda}+\frac{(\mathrm{i}\lambda)^{(1+s_0)/2}}{\mathrm{i}\lambda+1}ss_0\right)K_{\mathrm{int}}(|x|+|x_0|,(\lambda^2+1)^{-1}), \tag{17}$$

where

$$K_0(v)=\frac{1}{\sqrt{2\pi\,\mathrm{i}t}}\exp\left[\frac{\mathrm{i}}{2t}v^2\right],$$

$$K_{\mathrm{int}}(v,\nu)=-\frac{\nu}{2}\exp\left[\frac{\mathrm{i}t}{2}\nu^2+\nu v\right]\mathrm{erfc}\left[\sqrt{\frac{\mathrm{i}t}{2}}\nu+\frac{1}{\sqrt{2\,\mathrm{i}t}}v\right]. \tag{18}$$

As it should be, the above expressions for the δ and δ' agree with those calculated in the literature [14, 32, 33].

3. Wave packet scattering

Having discussed the full analytical solutions for arbitrary point interactions, we next shall present a physical picture to classify such potentials. To do so, we consider their different features in scattering off wave packets.

3.1. Wave packet time evolution

The time evolution of an arbitrary initial state $\Psi(x,0)$ is given by

$$\Psi(x,t)=\int_{-\infty}^{+\infty}\mathrm{d}x_0K(x,t;x_0,0)\Psi(x_0,0). \tag{19}$$

Thus, for each particular point interaction, one must consider the corresponding exact propagator from (14) and (15), and then solve (19) for a certain $\Psi(x_0, 0)$. In most cases the resulting integrals do not have analytical solutions, so numerics are necessary.

However, $\Psi(x, t)$ for point interactions can be calculated from rather simpler integrals. Indeed, in [18] it has been derived expressions which give the scattering solutions of a wave packet interacting with a general compact support potential, i.e., a potential of any arbitrary shape but identically null outside a certain region $x_l < x < x_r$. The only restriction on the derivations is that $\Psi(x, 0)$ must be well localized on one side of the potential, not overlapping significantly with it, i.e., $\Psi(x_l < x < x_r, 0) \approx 0$. Such expressions have been tested numerically [18], leading to a remarkably good agreement with the exact formula (19).

Thus, take the Fourier representation for the initial state

$$\Psi(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\, g(k) \exp[\mathrm{i}kx]. \tag{20}$$

If the above wave packet is well localized either to the left (+) or to the right (−) of the point interaction at $x = 0$, then from the amplitudes (3) we have that [18]

$$\begin{aligned}\Psi(x, t) &= \Psi^{(\mathrm{free})}(x, t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\, g(k) \left(\frac{c \pm \mathrm{i}k(d - a) + bk^2}{-c + \mathrm{i}k(d + a) + bk^2} \right) \\ &\quad \times \exp\left[\mp \mathrm{i}kx - \mathrm{i}\frac{k^2}{2}t \right] \qquad \text{for} \quad x \gtrless 0, \\ \Psi(x, t) &= \frac{\omega^{\pm 1}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k\, g(k) \left(\frac{2\,\mathrm{i}k}{-c + \mathrm{i}k(d + a) + bk^2} \right) \exp\left[\pm \mathrm{i}kx - \mathrm{i}\frac{k^2}{2}t \right] \quad \text{for} \quad x \lessgtr 0.\end{aligned} \tag{21}$$

Here $\Psi^{(\mathrm{free})}(x, t)$ represents the evolution of $\Psi(x, 0)$ in the absence of a potential. The advantage of equation (21) relies on the fact that we can study and classify how different point interactions scatter wave packets by directly analysing the behaviour of their reflection and transmission quantum amplitudes.

Important simple relations are derived from (21), see [18]. For instance, $P_T = \pm \int_0^{\pm\infty} \mathrm{d}k |g(k)|^2 |T^{(\pm)}(k)|^2$ gives how much of the initial wave packet localized to the left/right of the point interaction is transmitted through the potential after an infinite long time. Moreover, the corresponding average moment of the transmitted wave packet is $\langle p \rangle = \pm \int_0^{\pm\infty} \mathrm{d}k\, k |g(k)|^2 |T^{(\pm)}(k)|^2 / P_T$. Expressions for the reflected part of the wavefunction are similar. Note that these formulae depend only on the modulus square of the quantum amplitudes, which satisfy $|T^{(+)}|^2 = |T^{(-)}|^2$ and $|R^{(+)}|^2 = |R^{(-)}|^2$.

3.2. The potential parameter values and the scattering properties

Let us start with the δ and δ'. We have $T^{(\delta)} = \mathrm{i}k/(-\lambda + \mathrm{i}k)$, $R^{(\delta)} = \lambda/(-\lambda + \mathrm{i}k)]$, and $T^{(\delta')} = \lambda^{-1}/(\lambda^{-1} - \mathrm{i}k)$, $R^{(\delta')} = -\mathrm{i}k/(\lambda^{-1} - \mathrm{i}k)]$; thus

$$\begin{aligned} P_T^{(\delta)} &= \int_0^{+\infty} \mathrm{d}k |g(k)|^2 \frac{k^2}{\lambda^2 + k^2}, & \langle p \rangle^{(\delta)} &= \frac{1}{P_T^{(\delta)}} \int_0^{+\infty} \mathrm{d}k\, k |g(k)|^2 \frac{k^2}{\lambda^2 + k^2} \\ P_T^{(\delta')} &= \int_0^{+\infty} \mathrm{d}k |g(k)|^2 \frac{\lambda^{-2}}{\lambda^{-2} + k^2}, & \langle p \rangle^{(\delta')} &= \frac{1}{P_T^{(\delta')}} \int_0^{+\infty} \mathrm{d}k\, k |g(k)|^2 \frac{\lambda^{-2}}{\lambda^{-2} + k^2}. \end{aligned} \tag{22}$$

Suppose $g(k)$ is a function peaked at some value k_0. From (22) we find that larger values of k_0 lead to an increasing (decreasing) fraction of the initial wave packet tunnelling the δ (δ') potential. So, the δ' interaction has a characteristic opposite to the δ and to usual barrier

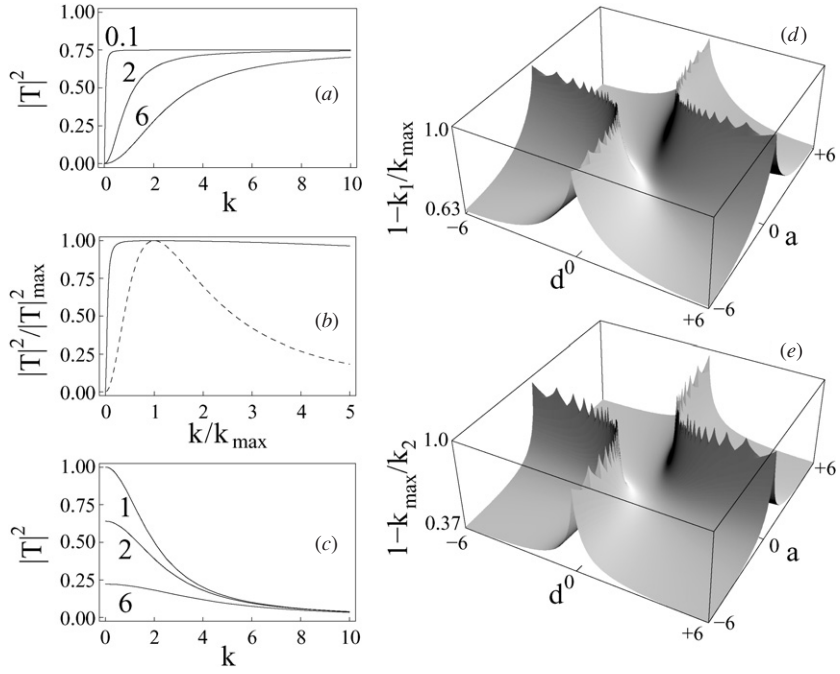

Figure 1. The transmission probabilities for the parameters: (*a*) $b = 0, d = 1.732\,05$ and three different values for c; (*b*) $b = 1$ and $a = 2d = 4$ (dashed curve), $a = 0.495, d = 2$ (continuous curve); and (*c*) $b = 1, a = 1/d$ and three different values for d. For $b = 1$ the rescaled inflection points k_1 and k_2 are shown, respectively, in (*d*) and (*e*).

potentials in quantum mechanics (see, e.g., [34]). It indicates that we can separate the point interactions into two distinct groups regarding their transmission properties. But recall that δ corresponds to $b = 0$ and δ' to $b \neq 0$. Also, from section 2.2 we see that b is the most relevant parameter to classify the analytical expressions for the K's. Thus, we have here a first link between the previous mathematical results and a physical feature for the potentials.

To extend the analyses, consider from (3) the transmission probabilities

$$\begin{aligned} |T(k, b = 0)|^2 &= \frac{4k^2}{c^2 + (d + d^{-1})^2 k^2}, \\ |T(k, b \neq 0)|^2 &= \frac{4b^2k^2}{(1 - ad)^2 + (2 + a^2 + d^2)b^2k^2 + b^4k^4}. \end{aligned} \tag{23}$$

For $b = 0$, if $d, c \neq 0$ then $|T(k)|^2$ increases monotonically from zero (at $k = 0$) to its asymptotic value of $4/(d + d^{-1})^2$, which assumes a maximum of 1 only when $d = \pm 1$ ($d = 1$ is the δ interaction). Note also the symmetry regarding $d \leftrightarrow 1/d$. In figure 1(*a*) we show this case for three different values of c and $d = 1.732\,05$ (then $4/(d+d^{-1})^2 = 0.75$). Other values of d display the same behaviour. For $b \neq 0$, $|T|^2$ has a maximum for $k = k_{\max} = \sqrt{|1 - ad|}/|b|$, so that

$$|T(k_{\max})|^2 \equiv |T|^2_{\max} = \frac{4}{2 + a^2 + d^2 + 2|1 - ad|}. \tag{24}$$

Also, in the expression for $|T|^2, k$ always appears multiplied by the parameter b. Thus, we can analyse the transmission probability curve setting $b = 1$ and keeping in mind that other values for b have as effect just to 'rescale' the wave number, i.e., to rescale the k-axis. If

$ad \neq 1$, $|T(k)|^2$ is null at $k = 0$, increases to its largest value at $k = k_{\max}$ and then starts to decrease asymptotically. Two examples are given in figure 1(*b*). For $ad = 1$, the maximum is at $k = 0$ and $|T(k)|^2$ decreases monotonically, as displayed in figure 1(*c*) for three different values of d.

So, we have that for $b = 0$ or $b \neq 0$ and $ad = 1$ the possible behaviours of $|T|^2$ are basically those seen in figures 1(*a*) and (*c*). The remaining question is if the shape of $|T(k)|^2$ for $b \neq 0$ and $ad \neq 1$ can be qualitatively different from the two curves in figure 1(*b*), e.g., a very sharp increasing for small k's (like the continuous curve) but with a relatively fast decreasing for $k > k_{\max}$ (like the dashed curve). To answer to that, we observe that generally $|T|^2$ has two inflection points, $k_1 \leqslant k_{\max}$ and $k_2 \geqslant k_{\max}$. Such points are important because in terms of k: the first is a measure of how fast the probability transmission increases to its maximum value; and the second indicates how fast is its asymptotic decay. In figures 1(*d*) and (*e*) we plot, respectively, the rescaled $1 - k_1/k_{\max}$ and $1 - k_{\max}/k_2$ as functions of a and d for $b = 1$. We see that they are peaked in the neighbourhood of the curve $ad = 1$, quickly decreasing to constant values (numerically of 0.6374 and 0.371 95) in the other regions of the a–d plane. Hence, we conclude that there are just three different behaviours for $|T|^2$ when $b \neq 0$. If $ad = 1$, we have essentially the shapes shown in figure 1(*c*). Note that in this case there is only one inflection point $k_2 = (d^2 + 1)/(\sqrt{3}d)$. For the product ad close to 1, but not exactly the unit, we have a very rapid growth of the transmission probability from $k = 0$. Once reaching its maximum, $|T|^2$ decreases very slowly, continuous line of figure 1(*b*). This is a consequence of the small values of $k_1/k_{\max}$ and $k_{\max}/k_2$. Finally, for the other regions of the a–d plane we have a relatively concentrate peak for $|T|^2$ around $k = k_{\max}$ (the dashed curve of figure 1(*b*)) because both k_1 and k_2 are comparable to $k_{\max}$.

Summarizing, there is an important qualitative difference between $b = 0$, figure 1(*a*), and $b \neq 0$, figures 1(*b*) and (*c*), then supporting the distinct expressions for the propagators (14) and (15). Moreover, for $b \neq 0$, the case $ad = 1$ has the particular property of $|T(k = 0)|^2 \neq 0$ if $0 < |d| < \infty$. This is also manifested in the formula for K, once for $b \neq 0$ the propagator has a different analytical structure if $ad - 1$ is or is not equal to zero. Indeed, if $ad = 1$, always one of the $\kappa_\pm$ is null; therefore the sum over the κ's in (15) instead of two has a single term.

Now we address how an incident wave packet of mean wave number k_0 tunnels different point interactions (we focus on the transmitted case only, the reflected case follows in a similar way). Obviously, the answer will depend on how close is $k_{\max}$ to k_0 and how broad is the transmission 'window' of $|T|^2$ compared to the wave packet moments width Δk. For explicit calculations, hereafter we consider the initial Gaussian wave packet

$$\Psi(x, 0) = \frac{1}{\sqrt{(2\pi)^{1/2}\xi}} \exp\left[\mathrm{i}k_0 x - \frac{(x - x_0)^2}{4\xi^2}\right], \tag{25}$$

for which

$$g(k) = \sqrt{\frac{2\xi}{\sqrt{2\pi}}} \exp[-\xi^2(k - k_0)^2 - \mathrm{i}(k - k_0)x_0]. \tag{26}$$

Also, note $R^{(-)}$, $R^{(+)}$ and $T^{(-)}$, $T^{(+)}$ are interchangeable simply from $a \leftrightarrow d$ and $\omega \leftrightarrow 1/\omega$. Thus, without loss of generality we can consider $\psi(x, 0)$ localized only to the left of the point interaction. In the calculations we set $x_0 = -10$, $k_0 = 10$ and $\xi = 0.3$. The initial state ($t = 0$) and its free evolution (i.e., when there is no potential) for $t = 1$ are shown in figure 2. In the inset we plot $|g(k)|^2$, equation (26).

If $b \neq 0$, the highest value that the transmission probability can assume, $|T|^2_{\max}$ in (24), depends only on a and d, having a maximum of 1 when $d = a$ and $|d| \leqslant 1$ (see figure 3(*a*)).

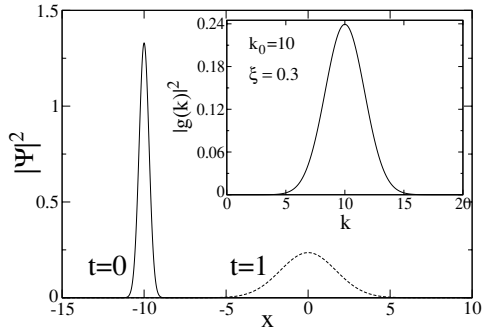

Figure 2. An initial ($t = 0$) Gaussian wave packet and its free evolution at $t = 1$. The inset represents the moments distribution of the initial state.

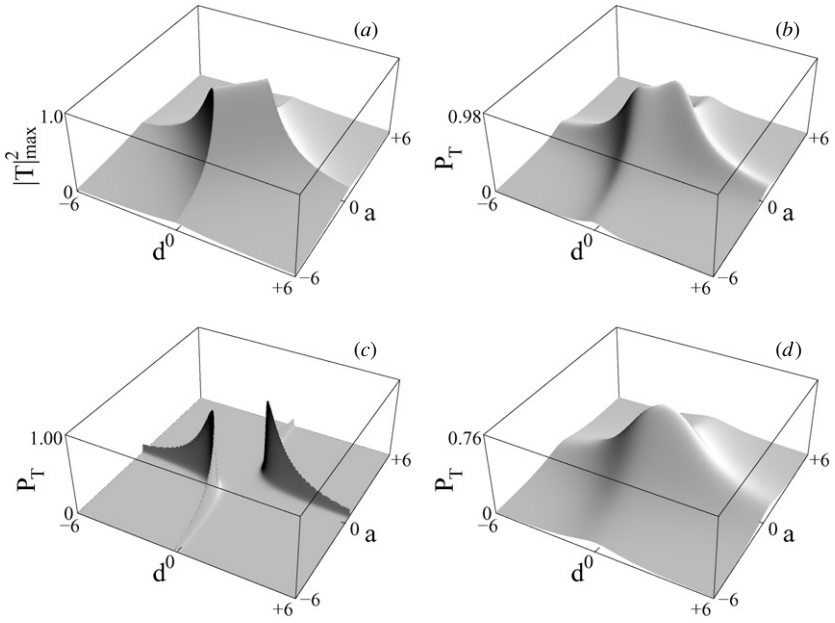

Figure 3. (*a*) The $|T|^2_{\max}$ dependence on a and d. The fraction P_T of the initial wave packet which tunnels the potential, as function of a and d, for b equal to b_1, b_2 and b_3 (see the text) is shown, respectively, in (*b*), (*c*) and (*d*).

On the other hand, $k_{\max}$ (which gives $|T|^2_{\max}$) is an explicit function of b. For a specific point interaction of defined $b \neq 0, a$ and d, $|T|^2_{\max}$ and $k_{\max}$ are then the two relevant quantities to determine how much of the initial wave packet will be transmitted through the potential. For instance, consider the three values of b: $b_1 = \sqrt{1 - 0.5 \times 0.5}/10$, $b_2 = \sqrt{1 - 1.5 \times 0.666}/10$, $b_3 = \sqrt{1 - 0.5 \times 0.5}/5$. The corresponding P_T's, as functions of a and d, are displayed in figures 3(*b*)–(*d*). For these b's, $k_{\max}$ coincides with the main moment component of the incident wave packet, $k_0 = 10$, for, respectively, $ad = 0.25$ or $ad = 1.75$, $ad = 0.999$ or $ad = 1.001$, and $ad = -2$ or $ad = 4$. For $b = b_1$, figure 3(*b*), the values of a and d for which $k_{\max} \approx k_0$ also result in $|T|^2_{\max}$ close to 1. It explains why in this case $P_T(a, b)$ has a shape similar to $|T|^2_{\max}(a, b)$ of figure 3(*a*). Since b_2 is relatively small, then $k_{\max} \approx k_0$ when $ad \approx 1$. For ad departing from this condition, $k_{\max}$ becomes large

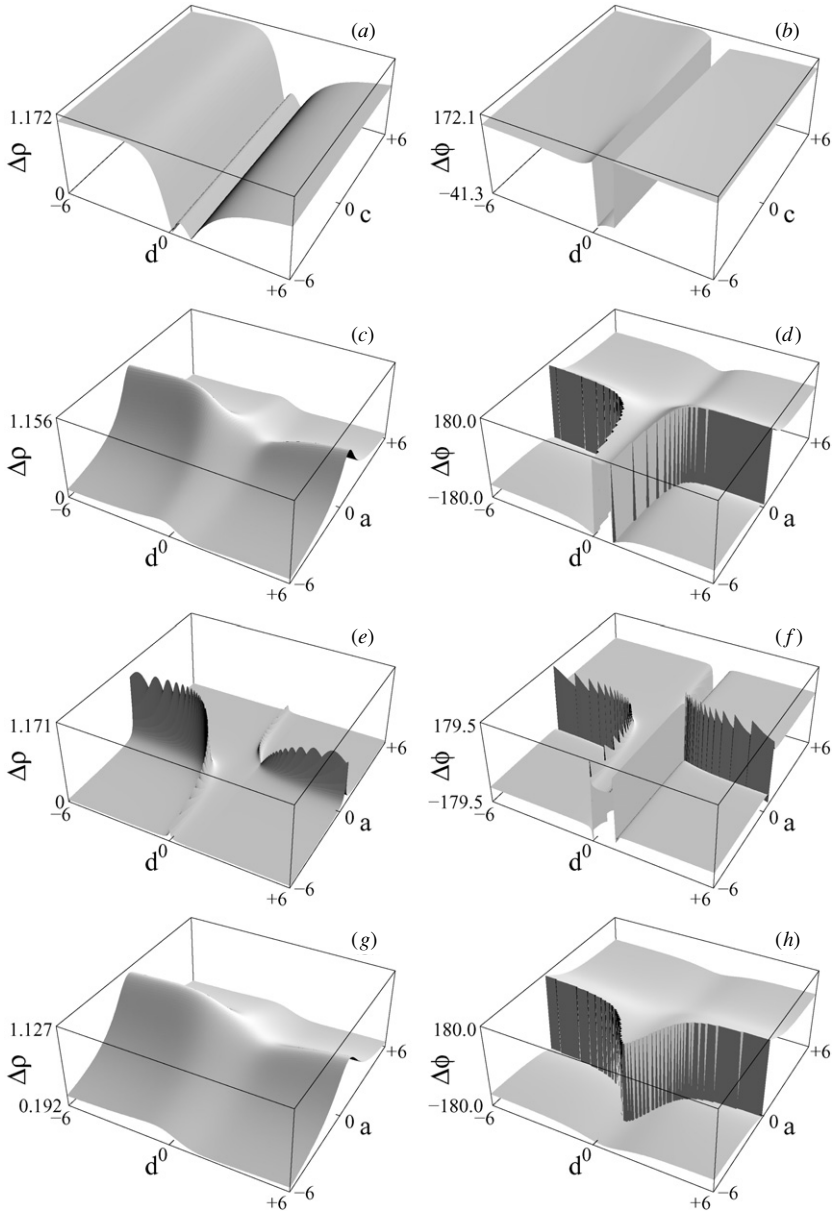

Figure 4. Plots at $t = 1$ of $\Delta\rho$ and $\Delta\phi$, where $\Delta\rho \exp[i\Delta\phi] \equiv \Psi(0^+) - \Psi(0^-)$, as functions of: d and c when $b = 0$, (a)–(b); a and d when $b = b_1$, (c)–(d), $b = b_2$, (e)–(f), and $b = b_3$, (g)–(h).

compared to k_0. But large k's do not contribute significantly to the initial wave packet (see $|g(k)|^2$ in figure 2). So, in figure 3(c) we have a highly concentrate P_T along $|1 - ad| \approx 1$ for $b = b_2$. Finally, consider $b = b_3$ and the set of a's and d's leading to $k_{\max} \approx k_0$. This set generates values for $|T|^2_{\max}$ which are not so close to 1 as in the case of $b = b_1$. Therefore, it follows the lower maximum and a more smooth and symmetric behaviour for P_T in figure 3(d) when compared to figure 3(b).

As a final interesting aspect of point interactions, we observe that their general mathematical definition, in terms of (1), connects the wavefunction (or its derivative) on

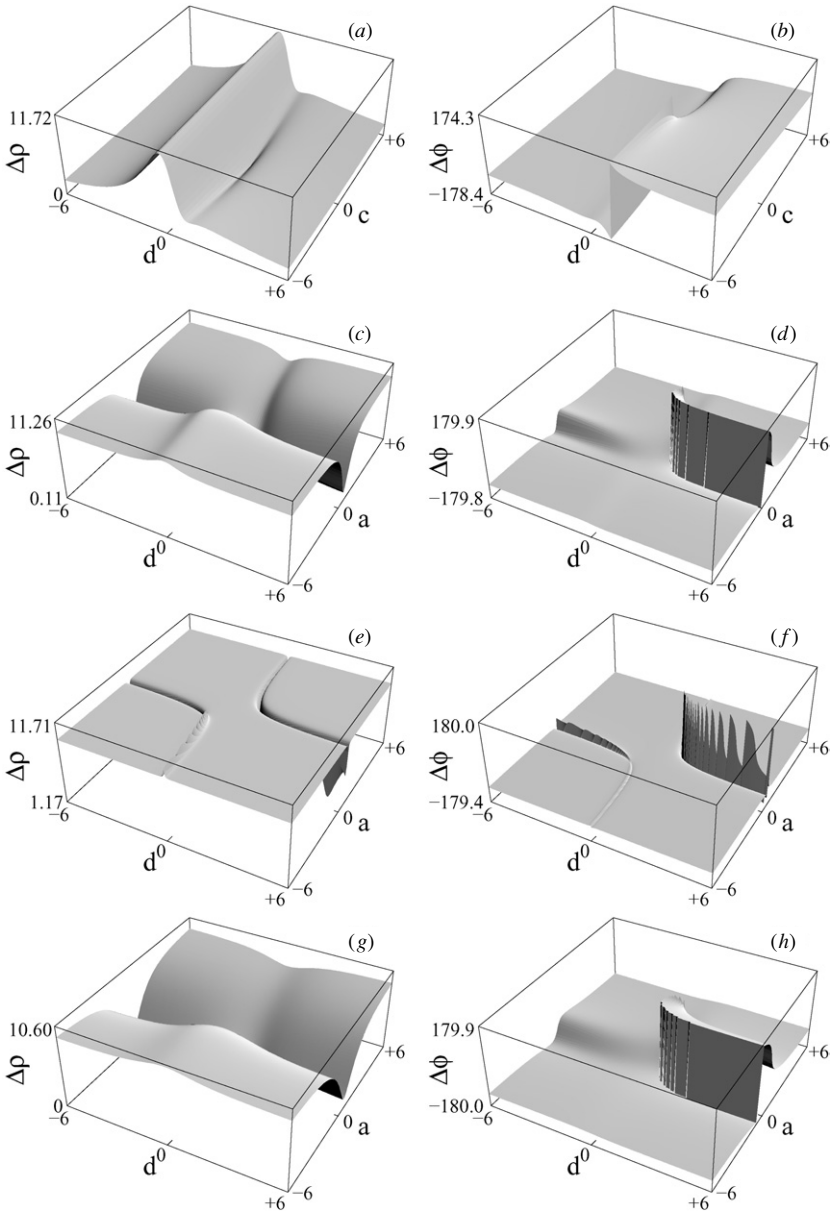

Figure 5. The same as figure 4, but for $\Delta\rho \exp[\mathrm{i}\Delta\phi] \equiv \Psi'(0^+) - \Psi'(0^-)$.

one side to *both* the wavefunction and its derivative on the other side of the zero-range potential. More usual in quantum mechanics, however, is to study the continuity, say across $x = 0$, of the wavefunction, by calculating $\Psi(0^+) - \Psi(0^-)$, and separately of the derivative, by calculating $\Psi'(0^+) - \Psi'(0^-)$.

To exemplify it, we consider our previous Gaussian wave packet and solve (21) and its spatial derivative numerically. For the calculations we choose $t = 1$, since it is the time necessary for the centre of the initial state to reach the position $x = 0$ in the free evolution case (see figure 2). For simplicity, we also set $\omega = 1$. Obviously, $\Psi(0^-)$ ($\Psi(0^+)$) is identified

with the first (second) relation in equation (21) for $x = 0$; likewise for $\Psi'(0^{\pm})$, which are the corresponding derivatives of (21) at $x = 0$. In figure 4 we show $\Delta\rho$ and $\Delta\phi$, where $\Delta\rho \exp[i\Delta\phi] \equiv \Psi(0^+) - \Psi(0^-)$, for different values of b, namely, 0, b_1, b_2 and b_3. Similar plots, but for $\Delta\rho \exp[i\Delta\phi] \equiv \Psi'(0^+) - \Psi'(0^-)$, are presented in figure 5. For convenience, $-180 \leqslant \Delta\phi \leqslant +180$ is measured in degrees. Just as a comparison, if $x = 0$ and $t = 1$ then $|\Psi^{(\mathrm{free})}| = 0.485$, $\mathrm{Arg}[\Psi^{(\mathrm{free})}] = -24.687$, $|\Psi^{(\mathrm{free})'}| = 4.854$ and $\mathrm{Arg}[\Psi^{(\mathrm{free})'}] = 65.313$.

Figures 4 and 5 give us then a glance of how diverse can be the behaviour of $\Psi(0^+) - \Psi(0^-)$ and $\Psi'(0^+) - \Psi'(0^-)$ for different point interactions. For instance, if $b = 0$, cases (a) and (b) in the figures, we find from (1) that $\Psi(0^+) - \Psi(0^-) = (d^{-1} - 1)\Psi(0^-)$ and $\Psi'(0^+) - \Psi'(0^-) = c\Psi(0^-) + (d - 1)\Psi'(0^-)$. For $d = 1$ (the δ function), as expected the wavefunction is continuous, so $\Delta\rho$ vanishes for any c (figure 4(a)). However, the derivative is not continuous, as seen in figure 5(a). For $d = 0$, the point interaction acts as an infinite wall and so $\Psi(0^+) = \Psi(0^-) = 0$, again leading to $\Delta\rho = 0$ in figure 4(a). Moreover, in this case $\Psi'(0^+) = c\Psi(0^-) = 0$ and thus the identity $\Psi'(0^+) - \Psi'(0^-) = -\Psi'(0^-)$ is not necessarily null, as can be verified from figure 5(a).

4. Conclusion

In this contribution we have presented the complete exact solutions, wavefunctions, Green functions and propagators, for the whole family of point interactions. We have derived very compact expressions for the propagators, which do summarize all the previous results in the literature. Also, we have discussed, mathematically, how to classify the potentials in terms of their parameter values, leading to different functional analytical formulae for the propagators.

Then, we have proposed a possible physical interpretation for such more formal results by studying how different point interactions scatter off wave packets. We have focused mainly on the transmission case once the reflection follows similarly. For our purposes we have considered only initial states well localized to one side of the potentials. So, for the calculations we have used relation (21), which is a very good approximation for the wave packet time evolution exact expression. The advantage of (21) is to allow a very straightforward way to analyse the different scattering properties of different point interactions.

We hope this work will be helpful to those considering point interactions as possible models to study varying phenomena. Since we propose a physical classification for point interactions, our results may serve as guide in selecting the proper values of the potential parameters to give the desired features in specific applications.

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Appendix

Here we separate the calculations for the propagators into four cases. However, we observe that afterwards the resulting expressions can be further compacted, leading to (14) and (15).

For $b = 0$, the case of x and x_0 on the same side

We have from (10)

$$K(x,t;x_0,0) = K_0(x,t;x_0,0) + \left(\frac{d^2-1}{d^2+1}\right)\frac{s_0}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k\,\exp[-\mathrm{i}ks_0(x+x_0)]\exp\left[-\mathrm{i}\frac{k^2}{2}t\right] + \left(\frac{2d^{(1+s_0)}\kappa_0}{d^2+1}\right)\theta(\kappa_0)\exp[\kappa_0 s_0(x+x_0)]\exp\left[\mathrm{i}\frac{\kappa_0^2}{2}t\right] + \left(\frac{2d^{(1+s_0)}\kappa_0}{d^2+1}\right)\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\,\frac{1}{(k-\mathrm{i}\kappa_0)}\exp[-\mathrm{i}ks_0(x+x_0)]\exp\left[-\mathrm{i}\frac{k^2}{2}t\right], \tag{A.1}$$

where K_0 is the propagator for the free particle (see section 2.3). Note that only the last two terms depend on the signal of κ_0. Furthermore, the integral involving $1/(k-\mathrm{i}\kappa_0)$, which we call $\mathcal{I}$, can be trivially cast in the form (11), so that

$$\mathcal{I} = -\left(\frac{2d^{(1+s_0)}\kappa_0}{d^2+1}\right)\theta(\kappa_0)\exp[\kappa_0 s_0(x+x_0)]\exp\left[\mathrm{i}\frac{\kappa_0^2}{2}t\right] + \left(\frac{d^{(1+s_0)}\kappa_0}{d^2+1}\right)\exp[\kappa_0 s_0(x+x_0)]\exp\left[\mathrm{i}\frac{\kappa_0^2}{2}t\right]\mathrm{erfc}\left[-\sqrt{\frac{\mathrm{i}t}{2}}\kappa_0 - \frac{s_0(x+x_0)}{\sqrt{2\,\mathrm{i}t}}\right]. \tag{A.2}$$

Thus, the exact propagator leads to

$$K(x,t;x_0,0) = K_0(x,t;x_0,0) + \left(\frac{d^2-1}{d^2+1}\right)s_0K_0(x,t;-x_0,0) + \left(\frac{d^{(1+s_0)}\kappa_0}{d^2+1}\right) \times \exp[\kappa_0 s_0(x+x_0)]\exp\left[\mathrm{i}\frac{\kappa_0^2}{2}t\right]\mathrm{erfc}\left[-\sqrt{\frac{\mathrm{i}t}{2}}\kappa_0 - \frac{s_0(x+x_0)}{\sqrt{2\,\mathrm{i}t}}\right]. \tag{A.3}$$

From the above expression it is clear that the propagator for this case has the same functional form regardless of the value of κ_0, as previously stated.

For $b = 0$, the case of x and x_0 on opposite sides

Now

$$K(x,t;x_0,0) = \frac{2d\omega^{s_0}}{(d^2+1)}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\,\exp[\mathrm{i}ks_0(x-x_0)]\exp\left[-\mathrm{i}\frac{k^2}{2}t\right] + \frac{2d\omega^{s_0}\kappa_0}{(d^2+1)}\theta(\kappa_0)\exp[-\kappa_0 s_0(x-x_0)]\exp\left[\mathrm{i}\frac{\kappa_0^2}{2}t\right] + \frac{2d\omega^{s_0}\kappa_0}{(d^2+1)}\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\,\frac{1}{(k-\mathrm{i}\kappa_0)}\exp[\mathrm{i}ks_0(x-x_0)]\exp\left[-\mathrm{i}\frac{k^2}{2}t\right], \tag{A.4}$$

which from (13) results in

$$K(x,t;x_0,0) = \frac{2d\omega^{s_0}}{(d^2+1)}K_0(x,t;x_0,0) + \frac{d\omega^{s_0}\kappa_0}{(d^2+1)} \times \exp\left[-\kappa_0 s_0(x-x_0)\right]\exp\left[\mathrm{i}\frac{\kappa_0^2}{2}t\right]\mathrm{erfc}\left[-\sqrt{\frac{\mathrm{i}t}{2}}\kappa_0 + \frac{s_0(x-x_0)}{\sqrt{2\,\mathrm{i}t}}\right]. \tag{A.5}$$

Again we have a single formula either if the potential admits or not a bound state.

For $b \neq 0$, the case of x and x_0 on the same side

In this case

$$K(x,t;x_0,0) = K_0(x,t;x_0,0) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k \exp[-\mathrm{i}ks_0(x+x_0)] \exp\left[-\mathrm{i}\frac{k^2}{2}t\right] + \frac{2}{\sqrt{(d-a)^2+4}}\left\{ \sum_{\kappa=\kappa_+,\kappa_-} \frac{\kappa\sigma(\kappa)}{(a+b\kappa)^{s_0}}\theta(\kappa)\exp[\kappa s_0(x+x_0)]\exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right] + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k \sum_{\kappa=\kappa_+,\kappa_-} \frac{\kappa\sigma(\kappa)}{(a+b\kappa)^{s_0}}\frac{1}{(k-\mathrm{i}\kappa)} \times \exp[-\mathrm{i}ks_0(x+x_0)]\exp\left[-\mathrm{i}\frac{k^2}{2}t\right]\right\}, \tag{A.6}$$

which can be rewritten as

$$K(x,t;x_0,0) = K_0(x,t;x_0,0) + K_0(x,t;-x_0,0) + \frac{2}{\sqrt{(d-a)^2+4}} \sum_{\kappa=\kappa_+,\kappa_-} \frac{\kappa\sigma(\kappa)}{(a+b\kappa)^{s_0}} \times \left\{\theta(\kappa)\exp[\kappa s_0(x+x_0)]\exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right] + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\,\frac{1}{(k-\mathrm{i}\kappa)}\exp[-\mathrm{i}ks_0(x+x_0)]\exp\left[-\mathrm{i}\frac{k^2}{2}t\right]\right\}. \tag{A.7}$$

Finally, the exact propagator is given by

$$K(x,t;x_0,0) = K_0(x,t;x_0,0) + K_0(x,t;-x_0,0) + \frac{1}{\sqrt{(d-a)^2+4}} \sum_{\kappa=\kappa_+,\kappa_-} \frac{\kappa\sigma(\kappa)}{(a+b\kappa)^{s_0}} \times \exp[\kappa s_0(x+x_0)]\exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right]\operatorname{erfc}\left[-\sqrt{\frac{\mathrm{i}t}{2}}\kappa - \frac{s_0(x+x_0)}{\sqrt{2\,\mathrm{i}t}}\right]. \tag{A.8}$$

Note that the functional form of K is the same for any signal of the $\kappa_\pm$.

For $b \neq 0$, the case of x and x_0 on opposite sides

Here, we find from (10)

$$K(x,t;x_0,0) = \frac{2\omega^{s_0}}{\sqrt{(d-a)^2+4}} \sum_{\kappa=\kappa_+,\kappa_-} \kappa\sigma(\kappa)\left\{\theta(\kappa)\exp[-\kappa s_0(x-x_0)]\exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right] + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\,\frac{1}{(k-\mathrm{i}\kappa)}\exp[\mathrm{i}ks_0(x-x_0)]\exp\left[-\mathrm{i}\frac{k^2}{2}t\right]\right\}. \tag{A.9}$$

Finally, from (13) we get

$$K(x,t;x_0,0) = \frac{\omega^{s_0}}{\sqrt{(d-a)^2+4}} \sum_{\kappa=\kappa_+,\kappa_-} \kappa\sigma(\kappa) \times \exp[-\kappa s_0(x-x_0)]\exp\left[\mathrm{i}\frac{\kappa^2}{2}t\right]\operatorname{erfc}\left[-\sqrt{\frac{\mathrm{i}t}{2}}\kappa + \frac{s_0(x-x_0)}{\sqrt{2\,\mathrm{i}t}}\right], \tag{A.10}$$

once more a unique formula independent of the values of $\kappa_\pm$.

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